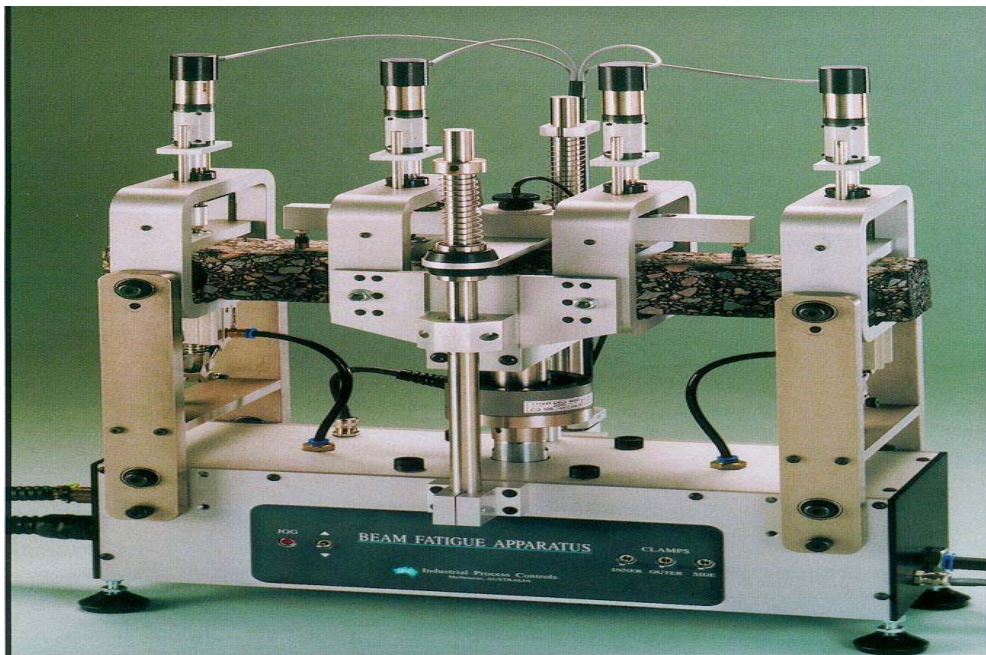# Theory of the

Four Point Dynamic

Bending test

Part II: Influences of

1. Overhanging Beam Ends
&
2. Extra Moving Masses

**Theory of the
Four Point Dynamic
Bending test**

**Part II: Influences of
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Date: August 2002

Abstract

This report deals with the influences of a) the extra moving masses and b) the overhanging beam ends on the measured deflections in the dynamic four point bending test.

The analytical solution is deduced for both cases. In the case of overhanging beam-ends the exact analytical solution can be obtained directly. However, if extra moving masses (point loads) are involved the numerical value for the analytical solution has to be obtained by an iteration procedure. In the included Excel files this value is obtained after nine iterations starting with the values obtained for the case with only overhanging beam-ends. Only when the chosen frequency is near the Eigen frequencies of the system more iteration will be necessarily.

After obtaining the correct values for the deflection and the phase lag at an arbitrarily position on the beam the modified first order approximation is applied for the backcalculation of the input values for the stiffness modulus and (material) phase lag. In this way an impression is obtained for the mistakes made in the backcalculation procedures.

Next to the chapters dealing with the theory a chapter "Guidelines" is given for the use of the included zipped Excel files.

Contents

The influence of extra moving masses and overhanging beam ends in the Four Point Bending Test

Introduction and Background

The basic theory of the four point bending test (4PB) is well described in part I of this series (Theory of the Four Point Dynamic bending test; P-DWW-96-008; ISBN-90-3693-712-4) and will not be repeated. Thus for a good understanding of the formulas it is advised to use part I next to part II. For a good description of the 4PB test is referred to: Description of fatigue test method Four Point Bending (4PB) in use at the Dutch Road & Hydraulic Engineering Division (DWW) & Technical University Delft & Dutch Consultancy Agencies; DWW-2002-082; July 2002. Photos of an old and a new clamping device are given below in figure 1 and 2.
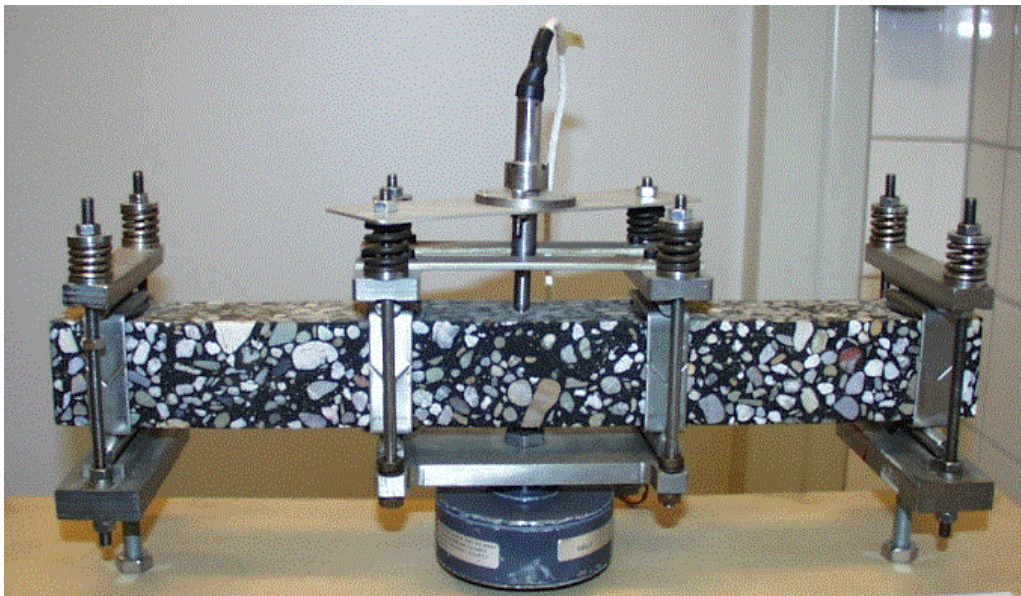
Figure 1. The "old" bending device.
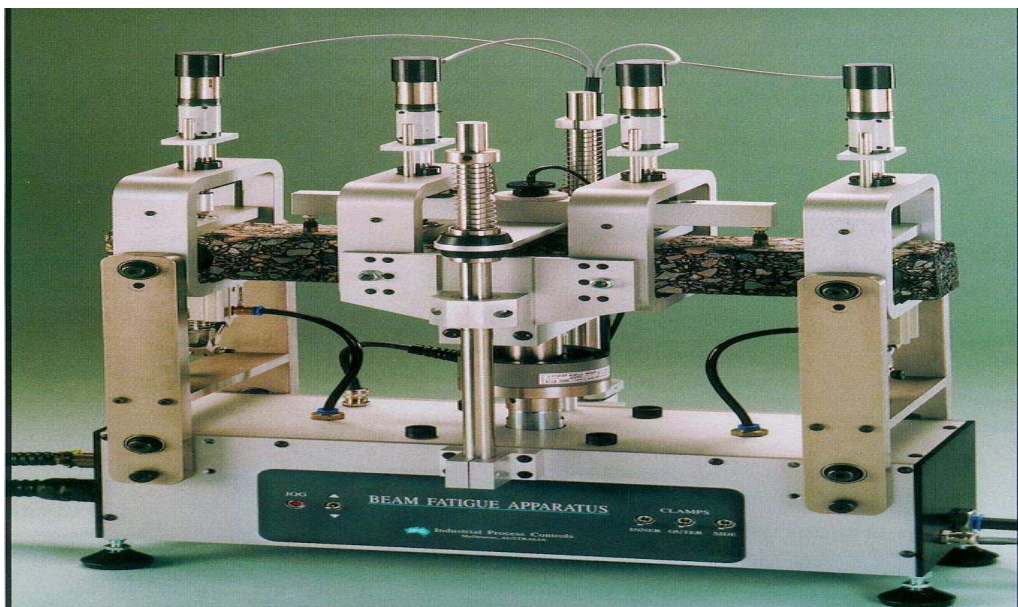
Figure 2. The new four point bending device

In the four point bending test the applied force will not only be used for bending the beam but also for the movements of parts of the hydraulic jack and of the inner clamps and if present the sensor which is placed on the beam for measuring the deflection. In the equations the influence of the mass of the sensor can be replaced by an arbitrarily mass at the position X_{sensor}. Thus the losses due to the movement of the masses have to be taken into account in the data processing. In the following equations the time function $\mathbf{e^{i.\omega_0 t}}$ has been omitted.

Theory

First of all we will use the following notations for the orthogonal polynomials used in the solution procedure (see P-DWW-96-008):

$$T_n\{x\} = Sin\left(n \times \pi \times \frac{x}{L^*}\right) \times \left[Sin\left(n \times \pi \times \frac{A+\Delta}{L^*}\right) - Sin\left(n \times \pi \times \frac{\Delta}{L^*}\right)\right] \tag{1}$$

$$T_n^*\{x\} = Sin\left(n \times \pi \times \frac{x}{L^*}\right) \times \left[Sin\left(n \times \pi \times \frac{x_{sensor}}{L^*}\right) - Sin\left(n \times \pi \times \frac{\Delta}{L^*}\right)\right] \tag{2}$$

With L^* = Total length of the beam; A = distance between inner and outer clamp and Δ = distance between beginning of beam (x=0) and the first outer clamp. The distance $A+\Delta$ is also called $\mathbf{X_{clamp}}$

The total deflection $\mathbf{V_{tot}\{x\}} = \mathbf{\hat{V}_{tot}}\{x\} \times \mathbf{e}^{-i.\varphi_{tot}^*\{x\}}$ has to satisfy the differential equation including the boundary conditions at the beginning of the beam (moment=0) and at the outer clamp (deflection=0). The deflection is build up out of the sum of three types of deflections.

The first type $\mathbf{V_a}\{\mathbf{x}\}$ will satisfy the differential equation including the external force $\mathbf{F_o}$ and the forces due to extra moving masses (particular solution). It is assumed that this deflection can be written as an infinite summation of the polynomials $\mathbf{T_n\{x\}}$ with n being an odd integer number. This is correct if only the external force and the force due to an extra moving mass at the inner clamps are present, because these forces can also be developed in the polynomials $\mathbf{T_n\{x\}}$. However, a mass force on a different position (e.g. a sensor in the middle of the beam) hampers in away such a simple solution procedure. In principle another deflection $\mathbf{V_a}\{\mathbf{x}\}$ has to be added to satisfy this mass force. This later deflection (in case of a mass at location X_{sensor}) is build up out of the polynomials $\mathbf{T_n^*\{x\}}$. Nevertheless, by rewriting the mass term it is possible to handle the problem with only the first deflection $\mathbf{V_a}\{\mathbf{x}\}$ type.

$$M_{sensor}.T_n^*\{x\} = \left[M_{sensor} \cdot \frac{T_n^*\{x\}}{T_n\{x\}}\right].T_n\{x\} = S_n.M_{sensor}.T_n\{x\} \tag{3}$$

The mass forces (point load) at the inner clamp and at the position of the sensor are given by:

$$\omega_0^2.M_{clamp}.V_{tot}\{X_{clamp}\} \quad \& \quad \omega_0^2.M_{sensor}.V_{tot}\{X_{sensor}\} \tag{4}$$

The deflection $\mathbf{V_a}\{\mathbf{x}\}$ is represented by:

5

$$V_a\{x\} = \sum V_{a,n}\{x\} = \sum \left[A_n \times e^{-i.\varphi_n^*} \times T_n\{x\} \right] = \hat{V}_a\{x\} \times e^{-i.\varphi_a\{x\}} \qquad [5]$$

The individual amplitude of each term at position **x** is given by: $\hat{V}_{a,n} = \left| A_n \times T_n\{x\} \right|$

By using the polynomials this deflection already satisfies the boundary condition at the beginning of the beam (x = 0 ; moment = 0). If Δ equals zero (outer clamp at position x=0) the boundary condition for the deflection (=0 at x=Δ) is also fulfilled. Furthermore, note that the phase lag φ_n^* for each separate deflection term doesn't depend on the position **x** along the beam; but because the polynomial $T_n\{x\}$ can change of sign along the beam for n values above 1, the final (observed/measured) phase lag (denoted as $\varphi_{a,n}^*$) may change along the beam. Due to the summation of all terms the resulting total phase lag φ_a^* (= φ_{tot}^* for Δ=0) will be a function of the position **x** along the beam.

If $\Delta > 0$ **t**he boundary condition (deflection = 0) at the outer clamp **(x=Δ)** is not yet fulfilled. In that case two other types of solutions $V_c\{x\}$ and $V_d\{x\}$ of the homogenous differential equation are needed.

The chosen types for the deflections $V_c\{x\}$ and $V_d\{x\}$ are solutions of the differential equation without the external force F_0 and the forces due to extra moving masses (although these last forces are related to the total deflections at those locations).

$$V_c\{x\} = \sum [C_n] \times Cos(f\{x\}); \quad V_d\{x\} = \sum [D_n] \times Cos(f\{x\} \times i) ;$$

$$f\{x\} = \beta_0 \times (x - \tfrac{L^*}{2}) \times e^{i.\alpha} \qquad [6]$$

The parameter α is equal to **-φ/4**.
It looks odd to use a summation sign because the function **f{x}** doesn't depend on the dummy integer **n** and both deflection functions can be written using only one constant:

$$C_{tot} = \sum [C_n] \ ; \ D_{tot} = \sum [D_n] \qquad [7]$$

However, using this summation makes it easier to understand how the boundary conditions can be fulfilled:
1) Each coefficient D_n is related to C_n by the fact that at the beginning of the beam **(x=0)** the moment (strain) has to be zero. This leads of course to the same relation for all **n** values. Mark that the deflection $V_a\{x\}$ already meets this requirement by the term:

$$Sin\left(n \times \pi \times \frac{x}{L^*} \right)$$

2) Afterwards each coefficient C_n can be determined by the requirement that at the outer clamp **(x=Δ)** the total deflection has to be zero. Which implies that for each term **n** the summation $V_{a,n} + V_{c,n} + V_{d,n}$ must be zero at this location:

$$\hat{V}_{tot,n}\{x = \Delta\} \times e^{-i.\varphi_{tot,n}\{x=\Delta\}} =$$

$$V_{a,n}\{x = \Delta\} + V_{c,n}\{x = \Delta\} + V_{d,n}\{x = \Delta\} = 0 \qquad [8]$$

For solving the differential equation the external force F_0 at the two inner clamps has to be transformed into a force distribution along the beam:

$$F_0 \Rightarrow \frac{2.F_0}{L^*} \times \sum \left[T_n\{x\} \right] \tag{9}$$

Using the same procedure for the extra mass forces at the two (moving) inner clamps (x=A+Δ) and at the position of the deflection sensor (x=x$_{sensor}$) will lead to:

$$\frac{Q_{Clamp}\{x\}}{\rho.b.h} = \omega_0^2 \times \frac{2 \times M_{Clamp}}{M_{beam}} \times \hat{V}_{tot}\{A + \Delta\} \times e^{-i.\left(\varphi^*\{A+\Delta\} \right)} \times \sum \left[T_n\{x\} \right] \tag{10}$$

$$\frac{Q_{sensor}\{x\}}{\rho.b.h} = \omega_0^2 \times \frac{2 \times M_{sensor}}{M_{beam}} \times \hat{V}_{tot}\{x_{sensor}\} \times e^{-i.\left(\varphi^*\{x_{sensor}\} \right)} \times \sum \left[T_n^*\{x\} \right] \tag{11}$$

The deflection term V_a must satisfy the differential equation with regard to the load conditions.

$$e^{i.\varphi} \times \frac{\partial^4}{\partial x^4}\left[V_a\{x\} \right] - \omega_0^2 \times V_a\{x\} - \frac{Q_{Clamp}\{x\}}{\rho \times b \times h} - \frac{Q_{sensor}\{x\}}{\rho \times b \times h} =$$

$$\frac{2.F_0}{M_{beam}} \times \sum \left[T_n\{x\} \right] \tag{12}$$

The deflection V_a can also be written as a summation of the polynomials $T_n\{x\}$. It follows the deflection V_a has to satisfy the next equation:

$$e^{i.\varphi} \times \sum \left[\xi_n^2 . V_{a,n}\{x\} \right] - \omega_0^2 \times \sum \left[V_{a,n}\{x\} \right] - \frac{Q_{Clamp}\{x\}}{\rho \times b \times h} - \frac{Q_{sensor}\{x\}}{\rho \times b \times h} =$$

$$\frac{2.F_0}{M_{beam}} \times \sum \left[T_n\{x\} \right] \tag{13}$$

With: $$\xi_n^2 = \frac{n^4 \times \pi^4}{L^{*4}} \times \frac{S_{mix} \times I}{\rho \times b \times h} \tag{14}$$

If no extra moving mass forces (point loads) are present closed analytical solutions are known. If these mass forces are present the solution has to be obtained by an iteration procedure. The iteration procedure will be outlined in the next paragraphs.
First of all the problem will be discussed in which the length L between the two outer clamps is equal to the actual length L^* of the beam (Δ=0).

A FIRST ORDER APPROXIMATION OF THE SOLUTION FOR Δ=0 FORMS IN FACT THE BASE OF THE FORMULAS USED IN BACKCALCULATION PROCEDURES FOR THE DETERMINATION OF STIFFNESS AND (MATERIAL) PHASE LAG FROM THE MEASURED DEFLECTION AND PHASE LAG BETWEEN FORCE AND DEFLECTION

The case Δ=0

If the beam length L^* equals the length L between the two outer clamps (Δ=0) a simplified procedure can be followed if the phase lag $\varphi_{tot}^*\{x\}$ doesn't vary (much) along the beam. The function $\mathbf{T_n\{x\}}$ alters into:

$$T_n\{x\} \Rightarrow P_n\{x\} = Sin\left(n \times \pi \times \frac{x}{L}\right) \times Sin\left(n \times \pi \times \frac{A}{L}\right) \qquad [15]$$

and $\mathbf{T_n^*\{x\}}$ alters into

$$T_n^*\{x\} \Rightarrow P_n^*\{x\} = Sin\left(n \times \pi \times \frac{x}{L}\right) \times Sin\left(n \times \pi \times \frac{x_{sensor}}{L}\right) \qquad [16]$$

Therefore the functions $\mathbf{V_c\{x\}}$ and $\mathbf{V_d\{x\}}$ are not needed to fulfil the boundary requirement at the outer clamp (x=Δ=0) and the remaining deflection $\mathbf{V_a\{x\}}$ satisfies already both boundary requirements and can be represented by:

$$V_{tot}\{x\} = V_a\{x\} = \sum\left[V_{a,n}\{x\} \right] = \sum\left[A_n \times e^{-i \cdot \varphi_n^*} \times P_n\{x\} \right] \qquad [17]$$

Defining $S_n^* = \dfrac{Sin\left(n \times \pi \times \dfrac{x_{sensor}}{L}\right)}{Sin\left(n \times \pi \times \dfrac{A}{L}\right)} \Rightarrow \sum\left[P_n^*\{x\}\right] = \sum\left[P_n\{x\} \times S_n^* \right] \qquad [18]$

it follows that each separate term has to fulfil the following equation:

$$e^{i \cdot \varphi} \times \xi_n^2 \times V_{a,n}\{x\} -$$

$$\omega_0^2 \times \left(V_{a,n}\{x\} + \left[\begin{array}{c} \dfrac{2 \times M_{Clamp}}{M_{beam}} \times V_a\{A\} \\[2mm] + \dfrac{2 \times M_{sensor}}{M_{beam}} \times V_a\{x_{sensor}\} \times S_n^* \end{array} \right] \times P_n\{x\} \right) \qquad [19]$$

$$= \frac{2.F_0}{M_{beam}} \times P_n\{x\}$$

The deflections at the positions of the extra moving masses $\mathbf{V_a\{ x_{clamp}=A+Δ =A\}}$ and $\mathbf{V_a\{x=x_{sensor}\}}$ are given by:

$$V_a\{x = A\} = \sum\left[V_{a,m}\{x = A\} \right] = \sum\left[A_m \times e^{-i \cdot \varphi_m^*} \times P_m\{A\} \right] \qquad [20]$$

$$V_a\{x = X_{sensor}\} = \sum V_{a,m}\{x = X_{sensor}\} = \sum A_m \times e^{-i \cdot \varphi_m^*} \times P_m\{X_{sensor}\} \qquad [21]$$

Dividing by the term $\mathbf{P_n\{x\}}$ leads to the following equation for each separate term:

$$e^{i \cdot \varphi} \times \xi_n^2 \times A_n \times e^{-i \cdot \varphi_n^*}$$

$$-\omega_0^2 \times \left(\begin{array}{l} A_n \times e^{-i \cdot \varphi_n^*} + \dfrac{2.M_{Clamp}}{M_{Beam}} \times \sum \left[A_m \times e^{-i \cdot \varphi_m^*} \times P_m\{A\} \right] + \\[3ex] \dfrac{2.M_{sensor}}{M_{beam}} \times \sum \left[A_m \times e^{-i \cdot \varphi_m^*} \times P_m\{x_{sensor}\} \right] \times S_n^* \end{array} \right) = \dfrac{2.F_0}{M_{beam}} \qquad [22]$$

The next move is to rewrite this equation in such a way that it will resemble the equation for a bending beam without extra moving masses.

$$e^{i \cdot \varphi} \times \xi_n^2 \times A_n \times e^{-i \cdot \varphi_n^*} - \omega_0^2 \times A_n \times e^{-i \cdot \varphi_n^*} \times$$

$$\left(\begin{array}{l} 1 + \dfrac{2.M_{Clamp}}{M_{Beam}} \times \sum_m \left[\dfrac{A_m}{A_n} \times e^{-i \cdot \left(\varphi_m^* - \varphi_n^* \right)} \times P_m\{A\} \right] + \\[4ex] \dfrac{2.M_{senor}}{M_{Beam}} \times \sum_m \left[\dfrac{A_m}{A_n} \times e^{-i \cdot \left(\varphi_m^* - \varphi_n^* \right)} \times P_m\{x_{sensor}\} \times S_n^* \right] \end{array} \right) = \dfrac{2.F_0}{M_{beam}} \qquad [23]$$

If the terms between the brackets are constants ($\varphi_m^* = \varphi_n^*$) the solution will be similar to the one for a bending beam without extra moving masses but with a heavier beam. The coefficients of the solution without extra moving masses are given by:

$$A_n = \frac{2 \times F_0 \times L^3}{\pi^4 \times S_{mix} \times I} \times \frac{1}{n^4 \times \sqrt{\left(1 - 2 \times Cos(\varphi) \times \dfrac{\omega_0^2}{\xi_n^2} + \dfrac{\omega_0^4}{\xi_n^4} \right)}} \qquad [24]$$

$$\varphi_n^* = Arctg \left[\frac{Sin(\varphi)}{Cos(\varphi) - \dfrac{\omega_0^2}{\xi_n^2}} \right] \qquad \text{(Note that the phase lags are in fact different)} \qquad [25]$$

$$\xi_n^2 = \frac{n^4 \times \pi^4}{L^4} \times \frac{S_{mix} \times I}{\rho \times b \times h} = \frac{n^4}{M_{Beam}} \times \frac{\pi^4 \times S_{mix} \times I}{L^3} \rightarrow \frac{L^3}{\pi^4 \times S_{mix} \times I} = \frac{n^4}{\xi_n^2 \times M_{Beam}}$$

Because the first term (n=1) represents already more than 90 % of the total deflection a first order approximation for the separate terms will be given by:

$$e^{i.\varphi} \times \xi_n^2 \times A_n \times e^{-i.\varphi_n^*} - \omega_0^2 \times A_n \times e^{-i.\varphi_n^*} \times$$

[26]

$$\left(1 + \frac{A_1}{A_n} \times \left[\frac{2.M_{Clamp}}{M_{Beam}} \times P_1\{A\} + \frac{2.M_{senor}}{M_{Beam}} \times P_1\{X_{sensor}\} \times S_1^*\right]\right) = \frac{2.F_0}{M_{beam}}$$

If the ratio A_1/A_n is taken equal to 1 this equation will be equal to the equation for a beam without extra moving masses if the term ω_0^2 is replaced by:

$$\omega_0^2 \Rightarrow \omega_0^2 \times \gamma_1 = \omega_0^2 \times \left(1 + \frac{2 \times M_{Clamp}}{M_{Beam}} \times P_1\{A\} + \frac{2 \times M_{sensor}}{M_{Beam}} \times P_1^*\{x_{sensor}\}\right)$$ [27]

> **The approximation above forms the base in the backcalculation procedure,**
>
> **which will be outlined later**

Simplified Iteration Procedure if Δ=0

The first order approximation can be used for the calculation of the deflection by using an iteration procedure as described in part I (P-DWW-96-008). It should be marked that this procedure is not used in the final calculation. The amplitude of the deflection at position X can be presented as:

$$\hat{V}_{tot}\{x\} = \hat{V}_a\{x\} = \left|\sum_{n=1}^{\infty}\left(A_n . e^{-i.\varphi_n^*\{x\}} P_n\{x\}\right)\right| \approx g_0 \times A_1 \times P_1\{x\} =$$

[28]

$$g_0 \times \frac{A_1}{A_n} \times A_n \times P_1\{x\} = g_0 \times g_n \times n^4 \times A_n \times P_1\{x\}$$

in which the coefficients g (and f) are close to one. The next step is the introduction of the parameter:

$$\gamma_n = 1 + n^4 \times \left(\begin{array}{l} g_0 \times g_n \times \dfrac{2 \times M_{clamp}}{M_{beam}} \times P_1\{A\} + \\[2mm] f_0 \times f_n \times \dfrac{2 \times M_{sensor}}{M_{beam}} \times P_1^*\{x_{sensor}\} \end{array}\right)$$

[29]

The starting values for the correction coefficients g_0, g_n, f_0 and f_n are 1. With these starting values the total deflections at $x = x_{clamp}$ and $x = x_{sensor}$ are calculated using equations 17, 24 and 25 and the transformation for the radial frequency given by equation 27.

The ratios of the total deflections at $x = x_{clamp}$ and $x = x_{sensor}$ and the first terms are used for a better approximation of g_0 and f_0. The coefficients g and f ought to be nearly the same.

$$g_0 = \hat{V}_a\{X_{clamp}\} / A_1 \times P_1\{X_{clamp}\} = \hat{V}_a\{X_{clamp}\} / \hat{V}_a\{X_{clamp,n=1}\}$$

[30]

$$f_0 = \hat{V}_a\{X_{sensor}\}\Big/ A_1 \times P_1\{X_{sensor}\} = \hat{V}_a\{X_{sensor}\}\Big/ \hat{V}_a\{X_{sensor,n=1}\}$$ [31]

The second steps are the calculations of 'all' the coefficients $A_n = \hat{V}_{a,n}\{X_{clamp}\}\big/ P_n\{X_{clamp}\}$

The same procedure is repeated for the deflections at $x = x_{sensor}$. The new values for A_n can be taken as a weighed mean value. The third step is the calculation of a better approximation for g_n and f_n.

$$g_n;f_n = \hat{V}_a\{X_{clamp};X_{sensor}\}\Big/\Big([g_0;f_0] \times n^4 \times A_n \times P_1\{X_{clamp};X_{sensor}\}\Big)$$ [32]

Of course g_1 and f_1 will be equal to 1.

These three steps are repeated until the coefficients and the deflections do not change anymore. The obtained figures for g_n, f_n and A_n can now be used for the calculation of the deflection along the whole beam.

General Iteration Procedure for $\Delta > 0$

For the general differential equation ($\Delta > 0$ and one or more extra moving masses) the following procedure can be used. First the deflection equation for each individual term is divided by the polynomial $T_n\{x\}$ leading to the following equation:

$$A_n.e^{-i.\varphi_n^*}.\left[e^{i.\varphi}.\xi_n^2 - \omega_0^2.\left(1 + \left[\begin{array}{c} \dfrac{2.M_{clamp}}{M_{beam}}.\dfrac{\hat{V}_{tot}\{x_{clamp}\}}{A_n}.\dfrac{e^{-i.\varphi_{tot}^*\{x_{clamp}\}}}{e^{-i.\varphi_n^*}} + \\[3mm] \dfrac{2.M_{sensor}}{M_{beam}}.\dfrac{\hat{V}_{tot}\{x_{sensor}\}}{A_n}.\dfrac{e^{-i.\varphi_{tot}^*\{x_{sensor}\}}}{e^{-i.\varphi_n^*}}.S_n \end{array} \right] \right) \right]$$ [40]
$$= \frac{2.F_0}{M_{beam}}$$

Rearrangement of this equation leads to: using former (old) values for A_n; φ_n^*; $\hat{V}_{tot}$; φ_{tot}^* for the terms between the brackets gives:

$$A_n.e^{-i.\varphi_n^*}.\left[e^{i.\varphi}.\xi_n^2 - \omega_0^2.\gamma_n.e^{+i.\phi_n} \right] = A_n.e^{-i.\psi_n^*}.\left[e^{i.(\varphi-\phi_n)}.\xi_n^2 - \omega_0^2.\gamma_n \right] = \frac{2.F_0}{M_{beam}}$$ [41]

with $\psi_n^* = \varphi_n^* - \phi_n$ and $\gamma_n = \left| \begin{array}{c} 1 + \dfrac{2.M_{clamp}}{M_{beam}}.\dfrac{\hat{V}_{tot}\{x_{clamp}\}}{A_n}.e^{-i.\left(\varphi_{tot}^*\{x_{clamp}\}-\varphi_n^* \right)} + \\[4mm] \dfrac{2.M_{sensor}}{M_{beam}}.\dfrac{\hat{V}_{tot}\{x_{sensor}\}}{A_n}.e^{-i.\left(\varphi_{tot}^*\{x_{sensor}\}-\varphi_n^* \right)}.S_n \end{array} \right|$ [42]

and

$$\phi_n = Argument \left\{ \begin{array}{l} 1 + \dfrac{2.M_{clamp}}{M_{beam}} \cdot \dfrac{\hat{V}_{tot}\{x_{clamp}\}}{A_n} \cdot e^{-i \cdot \left(\varphi_{tot}^*\{x_{clamp}\} - \varphi_n^* \right)} + \\[4mm] \dfrac{2.M_{sensor}}{M_{beam}} \cdot \dfrac{\hat{V}_{tot}\{x_{sensor}\}}{A_n} \cdot e^{-i \cdot \left(\varphi_{tot}^*\{x_{sensor}\} - \varphi_n^* \right)} \cdot S_n \end{array} \right\} \qquad [43]$$

The iteration process starts with the determination of the analytical solution for the bending of a beam without any extra moving masses. The step sequence is as follows:

1) From the equations 6, 7 and 8 for the boundary conditions the complex coefficients C_n and D_n are related to $A_n.e^{-i.\varphi_n^*}$. These last coefficients (particular solution) are at the start of the iteration process determined by the non-homogenous differential equation without the extra moving mass forces.
2) Calculate the total deflections V_{tot} at the locations X_{clamp} and X_{sensor}. using at least 25 n values.
3) Calculate for all n values the parameters γ_n and ϕ_n using equations 42 and 43.
4) Determine for n values new estimates for A_n and ψ_n^* using equation 41.
5) Calculate for all n values the new estimates for φ_n^*
6) Repeat step 1 to 5 but in step 1 the new estimates for A_n and φ_n^* from step 4 and 5 are used for the new estimates of C_n and D_n which are based on the boundary conditions for x = 0 and x = Δ. This sequence is repeated until the deflections in step 2 don't change significantly.

The main advantage of this approach is that equation 41 is similar to the one without extra moving mass forces. Although a lot of rewriting is needed this equation can be easily used in an iteration procedure.
Another faster and direct approach can be achieved by adding the extra moving mass forces to the applied external force. This leads to:

$$A_n.e^{-i.\varphi_n^*} \cdot \left[e^{i.\varphi} \cdot \xi_n^2 - \omega_0^2 \right] = \dfrac{2.F_0}{M_{beam}} \cdot \left[\begin{array}{l} 1 + \dfrac{M_{clamp}}{F_0} \cdot \hat{V}_{tot}\{x_{clamp}\} e^{-i.\varphi_{tot}^*\{x_{clamp}\}} + \\[4mm] \dfrac{M_{sensor}}{F_0} \cdot \hat{V}_{tot}\{x_{sensor}\} e^{-i.\varphi_{tot}^*\{x_{sensor}\}} . S_n \end{array} \right]$$

$$= \dfrac{2.F_0}{M_{beam}} \cdot Z_1 . e^{i.\theta_1} \qquad [43]$$

Rewriting $\left[e^{i.\varphi} \cdot \xi_n^2 - \omega_0^2 \right]$ as $Z_2 . e^{i.\theta_2}$ leads to the very simple iteration equation:

$$A_n.e^{-i.\varphi_n^*} = \dfrac{2.F_0}{M_{beam}} \cdot \dfrac{Z_1}{Z_2} . e^{i.(\theta_1 - \theta_2)} \qquad [44]$$

Both iteration procedures work fine. However, if the applied frequency is close to a resonance frequency the first procedure seems to be more stable.

Backcalculation procedure

First of all two new definitions R{x} and R'{x} are introduced:

$$R\{x\} = \frac{\pi^4}{2 \times P_1\{x\}} = \frac{\pi^4}{2 \times Sin\left(\pi \times \frac{A}{L}\right) \times Sin\left(\pi \times \frac{X}{L}\right)} \qquad [33]$$

$$R'\{x\} = \frac{12 \times L}{A} \times \frac{1}{3 \times \frac{X}{L} - 3 \times \frac{X^2}{L^2} - \frac{A^2}{L^2}} \qquad [34]$$

In fact R{x} is the coefficient used in the first term of the infinite sum which represents the analytical solution for a beam with Δ = 0. R'{x} is the mirror coefficient used in the solution of the static bending beam test.

The first order approximation for the deflection along the beam is given by:

$$V\{x,t\} \approx \frac{F_0 \times L^3}{R\{X\} \times S_{mix} \times I} \times \frac{e^{i.\left(\omega_0.t - \varphi_1^*\right)}}{\sqrt{1 - 2 \times Cos(\varphi) \times \frac{\omega_0^2}{\xi_1^2} \times \gamma_1 + \frac{\omega_0^4}{\xi_1^4} \times \gamma_1^2}} \qquad [35]$$

with $\gamma_1 = 1 + \frac{\pi^4}{R\{A\}} \times \frac{M_{clamp}}{M_{beam}} + \frac{\pi^4}{R\{X_{sensor}\}} \times \frac{R\{A\}}{R\{X_{sensor}\}} \times \frac{M_{sensor}}{M_{beam}}$ [36]

The deflection equation above is equal to the equation belonging to a viscous-elastic mass-spring system with a spring constant $\mathbf{K\{X\}} = \frac{\mathbf{S_{mix}} \times \mathbf{I}}{\mathbf{L^3}} \times \mathbf{R\{X\}}$ and an equivalent mass $\mathbf{M_{equi.}\{x\}}$:

$$M_{equi.}\{X\} = \frac{K\{X\}}{\xi_1^2} \times \gamma_1 = \frac{R\{X\}}{\pi^4} \times M_{beam} +$$

$$\frac{R\{X\}}{R\{A\}} \times M_{clamp} + \frac{R\{X\} \times R\{A\}}{\left[R\{X_{sensor}\}\right]^2} \times M_{sensor} \qquad [37]$$

Notice that both the spring constant and the equivalent mass depend on the position X at which the deflection is measured. Normally this will be at the location of the sensor, which in return is often placed in the centre of the beam (giving the highest amplitude). In that case the equivalent mass is given by equation 38.

$$M_{equi.}\left\{\frac{L}{2}\right\} = \frac{1}{2 \times Sin\left(\pi \times \frac{A}{L}\right)} \times M_{beam} +$$

$$Sin\left(\pi \times \frac{A}{L}\right) \times M_{clamp} + \frac{1}{Sin\left(\pi \times \frac{A}{L}\right)} \times M_{sensor}$$

[38]

Some equipments use the deflection measured at the inner clamp. In those cases the equivalent mass is given by:

$$M_{equi.}\{A\} = \frac{1}{2 \times Sin^2\left(\pi \times \frac{A}{L}\right)} \times M_{beam} +$$

$$M_{clamp} + \frac{Sin^2\left(\pi \times \frac{X_{sensor}}{L}\right)}{Sin^2\left(\pi \times \frac{A}{L}\right)} \times M_{sensor}$$

[39]

Instead of the function R{x} it is better to use the modified function R'{x} which gives better results in the backcalculation. Moreover, because for decreasing frequencies the equations become equal to the ones of a quasi-static system ($\omega_0 = 0$).

For the equipment at the Road & Hydraulic Eng. Div. of Rijkswaterstaat the following figures are valid: Effective length L of the beam = 400 mm, Distance A between inner and outer clamp = 135 mm. With these figures the following coefficients are obtained.

Using the deflection measured at the centre of the beam:
$M_{equivalent}$ = 0.5731 . M_{beam} + 0.8725 . M_{clamp} + 1.1461 . M_{sensor} (using R{x})
$M_{equivalent}$ = 0.5738 . M_{beam} + 0.8755 . M_{clamp} + 1.1423 . M_{sensor} (using R'{x})

Using the deflection measured at the inner clamp of the beam:
$M_{equivalent}$ = 0.6568 . M_{beam} + 1.0000 . M_{clamp} + 1.3136 . M_{centre} (using R{x})
$M_{equivalent}$ = 0.6555 . M_{beam} + 1.0000 . M_{clamp} + 1.3047 . M_{centre} (using R'{x})

In case of ASTM specifications (A=L/3) the following figures are obtained:

Using the deflection measured at the centre of the beam:
$M_{equivalent}$ = 0.5774 . M_{beam} + 0.8660 . M_{clamp} + 1.1547 . M_{sensor} (using R{x})
$M_{equivalent}$ = 0.5785 . M_{beam} + 0.8696 . M_{clamp} + 1.1500 . M_{sensor} (using R'{x})

Using the deflection measured at the inner clamp of the beam:
$M_{equivalent}$ = 0.6667 . M_{beam} + 1.0000 . M_{clamp} + 1.3333 . M_{centre} (using R{x})
$M_{equivalent}$ = 0.6652 . M_{beam} + 1.0000 . M_{clamp} + 1.3225 . M_{centre} (using R'{x})

Short Guidelines for the EXCEL files Visco1 and Visco2

Both Excel files consist out of three tabs: "Input-Output"; "Overhanging-Beam" and "Extra-Moving-Masses". The calculation procedures in both files are identical and differ only by the iteration procedure as described in the chapter "General Iteration Procedure for $\Delta > 0$". In the file Visco1.xls an extra option is given for the backcalculation if real measured deflections and phase lags are given (see below). The main tab is "Input-Output". The process starts with the input (bold red figures) for the:

Parameter	Cell	Dimension	Parameter	Cell	Dimension
Total length of beam	B2	[m]	Outer span Distance A	E2	[m]
Effective length of beam (span)	B3	[m]	Position X of calculation	E3	[m]
Height of beam	B4	[m]	Position sensor	E5	[m]
Width of beam	B5	[m]	The following parameters are related to the input and printed in blue italic characters:		
Mass of plunger	B6	[kg]			
Mass of sensor	B7	[kg]			
Mass of beam	B8	[kg]	- Density of the beam (B12);		
Stiffness Modulus	B10	[Pa]	- The bending Moment (B13);		
			- Overhanging Beam-end D (E4);		
Material phase lag	B11	[o]	- Position outer clamp (E6);		
			- Position inner clamp (E7).		
Force	B14	[N]			

With this input the static deflection can be calculated and is given in cell E18. The value for the frequency is put into cell C19. The calculated deflection and phase lag for position X (cell E3) are given in row 19 together with the backcalculated values for Smix, phase lag and strain using the calculated deflection and phase lag in combination with the modified first order approximation. An example is given on the next page in figure 3. It should be noticed that the exact analytical solution is build up out of an infinite series of sinusoidal terms. Because the accuracy of Excel calculations is limited to around 10^{-24} the summation is limited also because the values of higher terms decrease fast. Furthermore, the implication of this limitation is that a value of less than 10^{-20} can be considered to be zero.

In cell C19 the chosen frequency [Hz] is put in. The final output is given in the same row.

On tab "Overhanging-Beam" the calculations are given if only the mass of the beam was taken into account and not the extra moving masses due to plunger and sensor. The output figures are printed in the cells of block A5 to H8 (Figure 4.). In the cells of block I5 to J9 the output for the control calculations is given with respect to boundary requirements.

At last on tab "Extra-Moving-Masses" the calculations are performed if also the extra point loads at the position of the inner clamps are taken into account. As explained before the answers have to be obtained by iteration. An example is given in figure 5. The iteration procedure starts with the values obtained from the calculations on tab "Overhanging-Beam" when only the mass of the total beam is taken into account. In total nine iterations are performed. If the chosen frequency is not close to the Eigen frequency this number of iterations is more than satisfactorily. An approximation of the first Eigen frequency is given in the cells F4 and I4 on the tab "Input-Output".

	Columns				
A	**B**	**C**	**D**	**E**

A	B	C	D	E
Beam Characteristics			**Positions**	
Total length L* [m]	0.450		**Distance A (inner and outer clamp) [m]**	0.135
Effective length L [m]	0.400		**Xbeam [m] (X=0 :: begin of beam)**	0.225
Height H [m]	0.0503		**Delta**	0.025
Width W [m]	0.0628		**X Sensor [m] Default: = L*/2**	0.225
Mass plunger Mv [kg]	5.562		**Position outer clamp [m]**	0.025
Mass sensor [kg]	0.140		**Position inner clamp [m] = Position plunger mass**	0.160
Mass beam [kg]	3.254			
Material parameters				
Modulus Smix [Pa]	2.71E+09			
Phase lag φ [o]	35.2			
Density ρ [kg/m3]	2289			
Moment I [m4]	6.660E-07			
Force F [N]	52			
ξo	16			

Figure 3.
Input figures for tab "Input-Output"

Row 18/19	0	**Static Deflection [m]**	3.271E-05	**Smix [Pa] Backcalculated**	**φ [o] Back**	**φ [o] Measured**	**ε [m/m] Calculated**	**ε [m/m] Actual**	**Ratio ε**	**Ratio Smix**	**Ratio φ**
Frequency [Hz]	8	**"Dynamic" Deflection [m]**	3.302E-05	2.713E+09	35.255	35.623	4.896E-05	4.897E-05	100.0%	100.0%	100.0%

Position :	Xbeam	Xclamp	Xsensor	Position :	Xbeam	Xclamp	Xsensor	CONTROL	
Deflection [m]	3.280E-05	2.872E-05	3.280E-05	Smix back [Pa]	2.713E+09	2.713E+09	2.713E+09	*Deflection & Strain at X=Delta Deflection should be zero Deflection - Strain*	
φ [o]	35.35	35.35	35.35	φ back [o]	35.25	35.25	35.25	*2.69E-22*	3.01E-07
Strain [m/m]	4.862E-05	4.841E-05	4.862E-05	Strain Cal. [m/m]	4.863E-05	4.863E-05	4.863E-05	*Deflection & Strain at X=0 Strain should be zero Deflection - Strain*	
If only the mass of the beam is taken into account the results above are obtained								*6.41E-06*	*1.64E-23*

Figure 4
Example of output figures for tab "Overhanging-Beam"
including the control calculations for the boundary requirements

Deflection and Phase at Xbeam if all masses are taken into account

	Only Mass beam	First iteration	Second	Third	Fourth	Fifth	Sixth	Seventh	Eight	Nine		(It9-It8)/It9 [%]
Deflection [um]	32.8	33.0	33.0	33.0	33.0	33.0	33.0	33.0	33.0	33.0	Xbeam	0.000%
Phase lag [o]	35.35	35.62	35.62	35.62	35.62	35.62	35.62	35.62	35.62	35.62	Xbeam	0.000%
Control after 9 iterat.	**Deflection [m]** 3.302E-05		28.91	28.91	28.91	28.91	28.91	28.91	28.91	28.91	Xclamp	0.000%
Deflection at X=Δ	**φ meas. [o]** 35.62		35.62	35.62	35.62	35.62	35.62	35.62	35.62	35.62	Xclamp	0.000%
7.40E-21	**φ back [o]** 35.25				33.02	33.02	33.02	33.02	Xsensor	0.000%		
Strain at X=0	**Smix back [Pa]** 2.713E+09				35.62	35.62	35.62	35.62	Xsensor	0.000%		

		Xclamp	Xsensor
Deflection [m]		2.891E-05	3.302E-05
φ measured [o]		35.62	35.62
φ back. [o]		35.25	35.25
Smix back. [Pa]		2.713E+09	2.713E+09
Actual Strain [m/m]		4.866E-05	4.897E-05
Calc. Strain [m/m]		4.896E-05	4.896E-05

9.57E-22	**Actual Strain [m/m]**	4.897E-05
	Calc. Strain [m/m]	4.896E-05

DEFLECTION for Xbeam

Figure 5. Example of output figures for tab 'Extra-Moving-Masses

Special option in Visco1.xls for backcalculation

A special option in Visco1.xls allows the determination of the mistake made with the modified first order backcalculation procedure and the calculation of the 'correct' values. This is done in block A22 till D30 of tab Input-Output. The measured deflection and phase lag are put into cells B22 and B24. Seed values for the Smix and material phase lag are put into cells B10 and B11. The program calculates the deflection and phase lag, which are related to these input values (B23 and B25). Now in cell B26 the squared sum of the deviations between measured and calculated values is determined. Using the option Solver in Excel and varying the input values for Smix and phase lag can minimize this sum. The backcalculated Smix and phase lag values are given in the cells B28 and B30. In cells C28 and C30 the deviation percentage is given.

Deflection measured real	**3.2975E-05**		
Deflection calculated	3.3018E-05		
φ measured real	**35.623**		
φ calculated	35.6232		
Squared Summation	1.714E-06		*This value is minimized by the solver option*
Smix input	2.71E+09		
Smix back	2.71E+09	-0.03%	*Minimalization by variation in input data for Smix and φ*
φ input	35.244		
φ backcalculated	35.255	0.03%	

Figure 6. Example of the output for the special option in Visco1.xls

Remarks

1. In the Excel program file the phase angles are calculated as the argument of a complex number instead of using the arc tan function. This is done because the argument calculation gives the correct value between $-\pi$ and $+\pi$ while the arc tan function gives a value between $-\pi/2$ and $+\pi/2$. If the real part is denoted by a and the imaginary part by b the arc tan function only takes into account the sign of the quotient a/b while the argument function takes into account both the sign of a and the sign of b. The argument function is given by:

$$C.ARGUMENT(COMPLEX(a;b))$$

2. The absolute value of a complex figure z is calculated as:

$$C.ABS(z) \text{ or } C.ABS(COMPLEX(a;b)) \text{ or as } \sqrt{(a^2+b^2)}$$

3. The accuracy of the calculations in Excel is good but not extremely high. E.g. the constant pi is only given in 14 decimals. Therefore results smaller than 10^{-15} can be considered to be zero.

4. The two (free) Excel files can be ordered by sending an E-mail to:

A.C.Pronk@DWW.RWS.MINVENW.NL

5. An example is given below for a beam of 5 kg with a total length of 450 mm while the mass of the plunger is 10 kg and values for the stiffness modulus and the phase lag are 5 GPa and 20 $^{\circ}$ respectively.
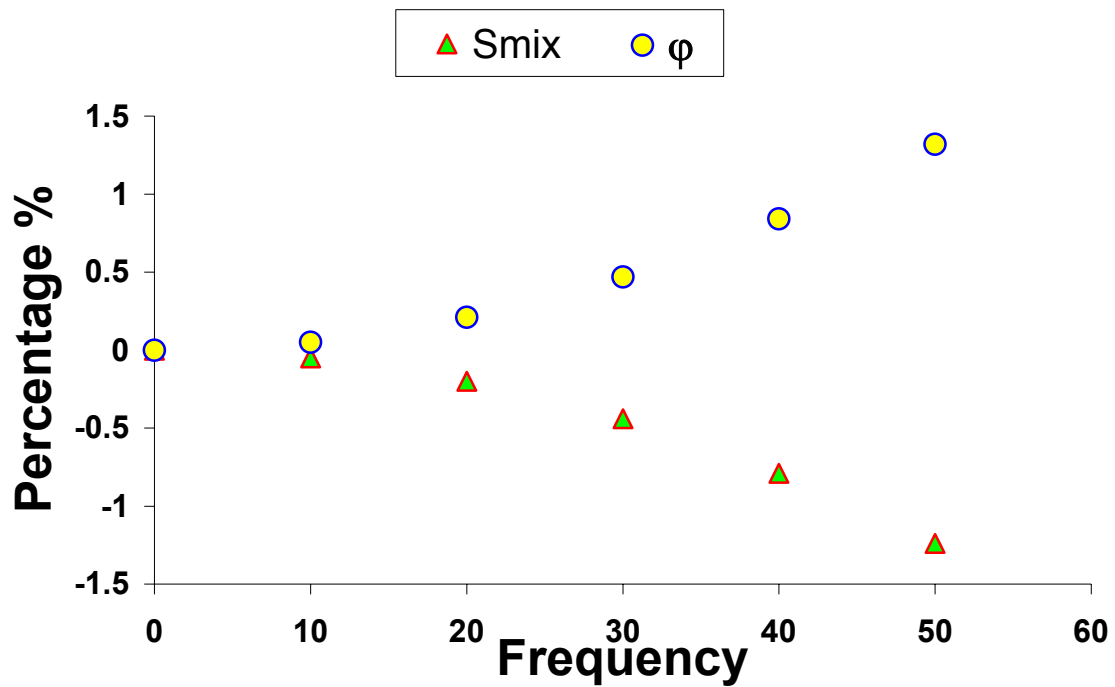
Figure 7. An example of the errors made in the backcalculation using the modified first order approximation